

Dimensions, lengths and separability in finite-dimensional quantum systems

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Many important sets of normalized states in a multipartite quantum system of finite dimension d , such as the set \mathcal{S} of all separable states, are real semialgebraic sets. We compute dimensions of many such sets in several low-dimensional systems. By using dimension arguments, we show that there exist separable states which are not convex combinations of d or less pure product states. For instance, such states exist in bipartite $M \otimes N$ systems when $(M-2)(N-2) > 1$. This solves an open problem proposed by DiVincenzo, Terhal and Thapliyal about 12 years ago. We prove that there exist a separable state ρ and a pure product state, whose mixture has smaller length than that of ρ . We show that any real $\rho \in \mathcal{S}$, which is invariant under all partial transpose operations, is a convex sum of real pure product states. In the case of the $2 \otimes N$ system, the number r of product states can be taken to be $r = \text{rank } \rho$. We also show that the general multipartite separability problem can be reduced to the case of real states. Regarding the separability problem, we propose two conjectures describing \mathcal{S} as a semialgebraic set, which may eventually lead to an analytic solution in some low-dimensional systems such as $2 \otimes 4$, $3 \otimes 3$ and $2 \otimes 2 \otimes 2$.

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I. INTRODUCTION

Entanglement reveals a fundamental difference between the quantum and classical world, which may be detected e.g. by Bell inequalities [2]. It plays an essential role in quantum information processing, such as quantum teleportation, computing and cryptography. It is a hard problem to decide whether a given quantum state is entangled. We shall propose a new method of attacking this problem in low-dimensional quantum systems, see Conjectures 4 and 5.

We consider a finite-dimensional multipartite quantum system described by the complex Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. We set $d = d_1 d_2 \cdots d_n$ where $d_i = \text{Dim } \mathcal{H}_i$. For brevity, we may refer to this system also as an $d_1 \otimes \cdots \otimes d_n$. A (*normalized*) *state* is a positive semidefinite linear operator $\rho : \mathcal{H} \rightarrow \mathcal{H}$ with $\text{Tr } \rho = 1$. If the condition $\text{Tr } \rho = 1$ is omitted and $\rho \neq 0$, we shall say that ρ is a *non-normalized state*. For convenience, we prefer to work with non-normalized states. We shall mention explicitly when we require the states to be normalized. A *pure product state* is the tensor product $|\psi_1\rangle\langle\psi_1| \otimes \cdots \otimes |\psi_n\rangle\langle\psi_n|$, where $|\psi_i\rangle \in \mathcal{H}_i$ are nonzero vectors. A state is *separable* if it is a finite sum of pure product states. A state is *entangled* if it is not separable. There exists a necessary and sufficient condition for separability [11] which shows that the separability problem is equivalent to another hard problem about positive linear maps. It has been shown that, in general, the problem of deciding whether a state is entangled is NP hard [8].

Let \mathcal{D} be the set of all normalized states on \mathcal{H} , and \mathcal{S} the subset of all normalized separable states. Let H denote the real vector space consisting of all Hermitian operators on \mathcal{H} . Denote by H' the affine subspace of H defined by the equation $\text{Tr } \rho = 1$. Our starting point is the well known fact that \mathcal{D} is a real semialgebraic subset of H' , see Section II. From now on we shall say “semialgebraic” instead of “real semialgebraic”. For the definition and examples of real algebraic and semialgebraic subsets of \mathbf{R}^n see e.g. [3, Section 2.1]. Let us just mention that $H_r := \{\rho \in H : \text{rank } \rho \leq r\}$ is a real algebraic set, and so is the set $H'_r = H' \cap H_r$. Since finite intersections of semialgebraic sets are semialgebraic, it follows that the sets $\mathcal{D}_r = \mathcal{D} \cap H_r$ are semialgebraic. We define the *length*, $L(\rho)$, of any separable state ρ to be the least integer r such that ρ can be written as a sum of r pure product states. In Proposition 2 we prove that the sets $\mathcal{S}'_r := \{\rho \in \mathcal{S} : L(\rho) \leq r\}$ are semialgebraic. From this result we deduce (see Corollary 3) that \mathcal{S} is a semialgebraic set, and so are the sets $\mathcal{S}_r = \mathcal{S} \cap H_r$.

Any closed semialgebraic set is a finite union of basic closed semialgebraic sets [3, Theorem 2.7.2]. Thus \mathcal{S} is a finite union of pieces, where each piece is described by a finite number of polynomial inequalities of the type $f(\rho) \geq 0$ with $f : H' \rightarrow \mathbf{R}$ a polynomial function. We conjecture that in the case of \mathcal{S} there is just one of these pieces, i.e., that \mathcal{S} is itself a basic closed semialgebraic set, see Definition 1 and Conjecture 4. A stronger version of this conjecture

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is stated as Conjecture 5. It asserts that the polynomial functions f in the inequalities $f(\rho) \geq 0$ defining \mathcal{S} can be chosen to be invariant under the action of the unitary groups of the \mathcal{H}_i . If the latter conjecture is true, then it is likely that the separability problem will be solved analytically in some low-dimensional cases where the Peres-Horodecki criterion fails.

We denote by Γ_i the partial transposition operator on the space \mathcal{H}_i , $i = 1, \dots, n$, computed in some fixed o.n. basis of that space. Thus, if $\rho = \rho_1 \otimes \dots \otimes \rho_i \otimes \dots \otimes \rho_n$ then $\rho^{\Gamma_i} = \rho_1 \otimes \dots \otimes \rho_i^T \otimes \dots \otimes \rho_n$. We denote by G the group generated by the pairwise commuting involutory operators Γ_i . The elements of G are the products $\Gamma_S = \prod_{i \in S} \Gamma_i$ where S runs through all subsets of $\{1, 2, \dots, n\}$. We say that a state ρ on \mathcal{H} has *positive partial transposes* (or that it is PPT) if $\rho^{\Gamma_S} \geq 0$ for all subsets S . If a state ρ is not PPT, we shall say that it is NPT. It is immediate from the definition of separable states that every separable state ρ is PPT.

We consider often the bipartite case ($n = 2$). In that case we set $\mathcal{H}_A = \mathcal{H}_1$, $\mathcal{H}_B = \mathcal{H}_2$, $M = d_1$ and $N = d_2$. We also set $\Gamma = \Gamma_1$. We say that a bipartite state ρ is a $k \times l$ state if its local ranks are k and l , i.e., $\text{rank } \rho_A = k$ and $\text{rank } \rho_B = l$ where $\rho_A = \text{Tr}_B(\rho)$ and $\rho_B = \text{Tr}_A(\rho)$. Let ρ be a $k \times l$ state. If $k = 1$ or $l = 1$ then ρ is clearly separable (and PPT). More generally, it is well known that ρ is separable if it is PPT and $(k-1)(l-1) \leq 2$ [11, 14].

Due to the fact that the sets \mathcal{D}_r , \mathcal{S}_r and \mathcal{S}'_r are semialgebraic, they have well defined dimensions. We have computed the dimension of \mathcal{D}_r for all $r = 1, \dots, d$, see Theorem 7. We also found tight upper bounds for $\text{Dim } \mathcal{S}'_r$. When these bounds do not exceed $d^2 - 1$, the dimension of the ambient affine space H' , they are saturated in most cases. For several low-dimensional quantum systems we have determined the $\text{Dim } \mathcal{S}'_r$ exactly, see Table I.

As a simple consequence of these bounds, we prove that if $(M-2)(N-2) > 1$ then there exist separable states on $M \otimes N$ of length exceeding MN . (E.g. in $3 \otimes 4$, there exist separable states of lengths 13 and 14.) This question was raised about 12 years ago [7] and remained open until now. In Conjecture 10, we propose a new candidate for the maximum length of separable states. (A counter-example to this conjecture was recently found by K-C. Ha and S-H. Kye [9].) Apparently, it is much harder to compute $\text{Dim } \mathcal{S}_r$. We were able to do that in $2 \otimes 2$ and $2 \otimes 3$ systems, see Proposition 11.

We investigate the set \mathcal{S}^{re} of separable density matrices $\rho \in \mathcal{S}$ all of whose entries are real. We denote by \mathcal{S}^G the set of G -invariant matrices $\rho \in \mathcal{S}$. If a Hermitian operator ρ is G -invariant, then $\rho^T = \rho$ and so $\rho^* = \rho$. i.e., ρ is real. Thus, we have $\mathcal{S}^G \subseteq \mathcal{S}^{\text{re}}$. In Proposition 13 we show that the set \mathcal{S}^G consists of all states ρ which admit a representation as a convex linear combination of normalized real pure product states. We denote by $L^G(\rho)$ the minimal number of real pure product states in such a convex linear combination. In Theorem 14 we prove that, in the $2 \otimes N$ system, we always have $L^G(\rho) = \text{rank } \rho$. The dimensions of the sets \mathcal{S}^{re} and \mathcal{S}^G are computed in Proposition 15. In analogy to the sets \mathcal{S}'_r , we introduce the sets $\mathcal{S}_r^G = \{\rho \in \mathcal{S} : L^G(\rho) \leq r\}$. For bipartite systems of dimension $d \leq 16$, we give in Table II the lower bounds for dimensions of the sets \mathcal{S}_r^G for $r = 1, \dots, d$. In most cases it is shown that the bound is actually equal to $\text{Dim } \mathcal{S}_r^G$. We consider the following three separability problems (S_1) , (S_2) and (S_3) : for a given state ρ , decide whether ρ belongs to \mathcal{S} , \mathcal{S}^{re} or \mathcal{S}^G , respectively. Since $\mathcal{S} \supseteq \mathcal{S}^{\text{re}} \supseteq \mathcal{S}^G$, (S_2) is a special case of (S_1) , and (S_3) a special case of (S_2) . We prove in Proposition 16 that (S_1) , (S_2) and (S_3) are equivalent to each other.

The content of the paper is as follows. In section II we prove that the sets \mathcal{D}_r , \mathcal{S}_r and \mathcal{S}'_r are semialgebraic and propose the two separability conjectures. In section III we compute $\text{Dim } \mathcal{D}_r$, find a tight upper bound for $\text{Dim } \mathcal{S}'_r$, and in some cases prove that they are equal. In section IV we characterize the set \mathcal{S}^G and compute its dimension. Finally, in section V we highlight some of our results and discuss the prospects of solving analytically the separability problem in some quantum systems of low dimension.

II. SOME SEMIALGEBRAIC SETS AND SEPARABILITY CONJECTURES

The sets \mathcal{D} , \mathcal{S} , \mathcal{D}_r , \mathcal{S}_r and \mathcal{S}'_r are compact subsets of H' , and the first two are also convex. It is not hard to prove that all of them are also semialgebraic subsets of H' .

Let us recall and show that \mathcal{D} is a semialgebraic subset of H' . Indeed, if $f(t) = \sum_{i=0}^d (-1)^i c_i t^{d-i}$, ($c_0 = 1$), is the characteristic polynomial of $\rho \in H'$, then $\rho \in \mathcal{D}$ if and only if each $c_i \geq 0$. Recall that c_i is the sum of all principal minors of order i of ρ , and so c_i is a polynomial function on H' . Thus, \mathcal{D} is semialgebraic. In fact this shows that \mathcal{D} is a basic closed semialgebraic set according to the following definition.

Definition 1 (see [3, Definition 2.7.1]) *A subset X of a Euclidean space E is a basic closed semialgebraic set if there exist finitely many real polynomial functions $f_i : E \rightarrow \mathbf{R}$, $i = 1, \dots, k$, such that $X = \{x \in E : f_i(x) \geq 0, i = 1, \dots, k\}$.*

(We shall use this definition when E is either H or H' .)

Since H_r is a real algebraic set and $\mathcal{D}_r = \mathcal{D} \cap H_r$, it follows that \mathcal{D}_r is semialgebraic. We shall now prove that this is also true for the sets \mathcal{S}'_r and \mathcal{S}_r .

Proposition 2 *Each set \mathcal{S}'_r is a semialgebraic subset of H' .*

Proof. Let X_i be the unit sphere in \mathcal{H}_i and let the subset $\Lambda \subset \mathbf{R}^r$ be defined by the equality $\lambda_1 + \dots + \lambda_r = 1$ and the inequalities $0 \leq \lambda_i \leq 1$ for $i \in \{1, \dots, r\}$. The spheres X_i are real algebraic sets and so is their product $X = X_1 \times \dots \times X_n$. As the set Λ is semialgebraic, the product

$$Z = \Lambda \times X^r \quad (1)$$

of Λ and r copies of X is also semialgebraic. We shall now define a map $f : Z \rightarrow \mathcal{S}'_r$. We shall write an arbitrary point $z \in Z$ as $z = (\lambda_1, \dots, \lambda_r, x_1, \dots, x_r)$, where each $x_i = (x_{i1}, \dots, x_{in}) \in X$. The function f is defined by setting $f(z) = \sum_{i=1}^r \lambda_i |x_i\rangle\langle x_i| \in \mathcal{S}'_r$, where $|x_i\rangle\langle x_i| = |x_{i1}\rangle\langle x_{i1}| \otimes \dots \otimes |x_{in}\rangle\langle x_{in}|$ is a pure product state. The assertion follows from [3, Theorem 2.8.8] since f is a real polynomial map and $f(Z) = \mathcal{S}'_r$. \square

Corollary 3 *The set \mathcal{S} of all normalized separable states on \mathcal{H} is semialgebraic, and so are its subsets \mathcal{S}_r .*

Proof. Since $\mathcal{S}_r = \mathcal{S} \cap \mathcal{D}_r$, the second assertion follows from the first. We shall give two proofs for the first assertion.

First proof: By a result of P. Horodecki [12, Theorem 1], which easily extends to the multipartite case, we have $\mathcal{S} = \mathcal{S}'_r$ for $r = d^2$ and the assertion follows from Proposition 2.

Second proof: By Proposition 2, \mathcal{S}_1 is semialgebraic. We can now apply the known fact that the convex hull of a semialgebraic set is also semialgebraic. As \mathcal{S} is the convex hull of \mathcal{S}_1 , we deduce that \mathcal{S} is semialgebraic. \square

Our first conjecture says that \mathcal{S} is a very nice semialgebraic subsets of H' .

Conjecture 4 *The set \mathcal{S} is a basic closed semialgebraic subset of H' .*

Our second conjecture is a stronger version of this one. It asserts that the polynomial functions, which occur in the representation of \mathcal{S} as a basic closed semialgebraic set, can be chosen in a very special way. In this place, it is convenient to work with the non-normalized states. If $X \subseteq H'$ is a nonempty subset, we define the *cone* over X to be the subset $KX = \{t\rho : t \geq 0, \rho \in X\}$ of H . Note that the vertex of KX , namely the origin of H , belongs to KX . Consequently, if X is closed and compact, then KX is closed. Note also that if X is convex or semialgebraic then KX has the same property, and conversely. We are in particular interested in the cone $K\mathcal{S}$ consisting of all non-normalized separable states (plus the origin).

The direct product of the general linear groups $\text{GL} := \prod_{i=1}^n \text{GL}_{d_i}(\mathbf{C})$ acts naturally on \mathcal{H} via local invertible transformations, and also acts on H . Explicitly, if $V \in \text{GL}$ and $\rho \in H$ then the latter action is given by $(V, \rho) \rightarrow V\rho V^\dagger$. The *local unitary group*, i.e., the subgroup $U = \prod_{i=1}^n U(d_i)$ of G , also acts on the same spaces. We say that a polynomial function $f : H \rightarrow \mathbf{R}$ is *invariant*, if $f(V\rho V^\dagger) = f(\rho)$ for all $V \in U$ and $\rho \in H$. The Conjecture 4 implies that the cone $K\mathcal{S}$ is a basic closed semialgebraic subset of H . The following conjecture is much stronger.

Conjecture 5 *There exist finitely many homogeneous invariant polynomial functions $f_i : H \rightarrow \mathbf{R}$, $i = 1, \dots, k$, such that $K\mathcal{S} = \{\rho \in H : f_i(\rho) \geq 0, i = 1, \dots, k\}$.*

Note that in the bipartite case both conjectures are true if $(M-1)(N-1) \leq 2$. In spite of the fact that this has been known for long time (albeit not stated in this way), all other bipartite cases still remain unsolved. If Conjecture 5 is true, then it should be possible to find analytic criteria of separability in some additional low-dimensional cases, say for $2 \otimes 4$, $3 \otimes 3$ and $2 \otimes 2 \otimes 2$ quantum systems. To realize this objective, it is first of all necessary to find a practical method for computing the homogeneous invariants of small degree. Since U is compact, the algebra of polynomial invariants is finitely generated and the generators can be chosen to be homogeneous. We pose the following problem which would provide a simple method for computing all homogeneous invariants.

Problem 6 *Find a minimal set of homogeneous generators for the algebra of invariant polynomials $f : H \rightarrow \mathbf{R}$ for the quantum systems $2 \otimes 4$, $3 \otimes 3$ and $2 \otimes 2 \otimes 2$.*

III. DIMENSION COMPUTATIONS FOR SOME QUANTUM SYSTEMS

Since all of the sets \mathcal{D}_r , \mathcal{S}'_r and \mathcal{S}_r are semialgebraic, they have a well defined notion of dimension. We shall compute some of these dimensions in certain cases. We start with the sets \mathcal{D}_r in which case we can ignore the tensor product structure of \mathcal{H} .

Theorem 7 *We have $\text{Dim } \mathcal{D}_r = r(2d - r) - 1$ for $r = 1, 2, \dots, d$.*

Proof. Any $\rho \in \mathcal{D}_r$ has a spectral decomposition

$$\rho = \sum_{i=1}^r \lambda_i |\psi_i\rangle\langle\psi_i|, \quad \sum \lambda_i = 1, \quad \lambda_i \geq 0, \quad \|\psi_i\| = 1. \quad (2)$$

We fix an o.n. basis $|\alpha_k\rangle$, $k = 1, \dots, d$ of \mathcal{H} and denote by $U(d)$ the global unitary group of \mathcal{H} with respect to this basis. This group acts naturally on \mathcal{H} as well as on H and H' and its subset \mathcal{D} . For $U \in U(d)$ and $\rho \in H$, we shall use the notation $U \cdot \rho = U\rho U^\dagger$. Denote by X the set of all states ρ given by (2) with $|\psi_i\rangle = |\alpha_i\rangle$, $i = 1, \dots, r$. Clearly, we have $\mathcal{D}_r = U(d) \cdot X$. Let $Y \subset X$ consist of all ρ which also satisfy the inequalities $\lambda_1 > \lambda_2 > \dots > \lambda_r$. The stabilizer in $U(d)$ of any state $\rho \in Y$ is the subgroup $U(1)^r \oplus U(d-r)$ of dimension $r + (d-r)^2$. Hence the dimension of the orbit $U(d) \cdot \rho$ is equal to $r(2d-r) - r$. Since $\text{Dim } Y = r-1$, we infer that $\text{Dim } U(d) \cdot Y = r(2d-r) - 1$. As the closure of $U(d) \cdot Y$ contains X , it must be equal to $U(d) \cdot X = \mathcal{D}_r$. Since $U(d) \cdot Y$ is a semialgebraic set, it follows that also $\text{Dim } \mathcal{D}_r = r(2d-r) - 1$. \square

Note that for $r = d$ we recover the well known fact that $\text{Dim } \mathcal{D} = \text{Dim } H' = d^2 - 1$.

Theorem 8 *For all positive integers r we have*

$$\text{Dim } \mathcal{S}'_r \leq r(1 + 2 \sum_i (d_i - 1)) - 1. \quad (3)$$

Consequently, there exist separable states of length

$$l := \left\lceil \frac{d^2}{1 + 2 \sum_i (d_i - 1)} \right\rceil, \quad (4)$$

where $\lceil t \rceil$ denotes the smallest integer $\geq t$.

Proof. The space Z (see Eq. (1)) and the map $f : Z \rightarrow \mathcal{S}'_r$ provide a parametrization of the set \mathcal{S}'_r . This parametrization is redundant in the sense that the overall phases of the unit vectors $|\alpha_i\rangle$ and $|\beta_i\rangle$ are irrelevant. To obtain a more economical parametrization of \mathcal{S}'_r , we replace each sphere X_i by its $(2d_i - 2)$ -dimensional subsphere, X_i^0 , consisting of all unit vectors whose first component is real. Let $X^0 = X_1^0 \times \dots \times X_n^0$ and $Z_0 = \Lambda \times (X^0)^r$, and let $f_0 : Z_0 \rightarrow \mathcal{S}'_r$ be the restriction of the map f used in the proof of Proposition 2. Since we still have $f_0(Z_0) = \mathcal{S}'_r$, it follows that $\text{Dim } \mathcal{S}'_r \leq \text{Dim } Z_0 = r(1 + 2 \sum_i (d_i - 1)) - 1$.

If $r = l - 1$ then $r(1 + 2 \sum_i (d_i - 1)) < d^2$, and so $\text{Dim } \mathcal{S}'_r < d^2 - 1 = \text{Dim } \mathcal{S}$. Thus \mathcal{S}'_r is a proper subset of \mathcal{S} , and so there exist $\rho \in \mathcal{S}$ with $L(\rho) \geq l$. \square

About 12 years ago the authors of [7] raised the question whether the separable states on $M \otimes N$ satisfy the inequality $L(\rho) \leq MN$. It follows from the theorem that the answer to this question is negative. Indeed, if $(M-2)(N-2) > 1$ then $\text{Dim } \mathcal{S} = M^2 N^2 - 1 > MN(2M + 2N - 3) - 1 \geq \text{Dim } \mathcal{S}'_{MN}$ and so there must exist separable states on \mathcal{H} of length bigger than MN .

It is now easy to answer the analogous question in the multipartite case.

Corollary 9 *Assume that $d_1 \geq d_2 \geq \dots \geq d_n \geq 2$ and $n \geq 2$. If $L(\rho) \leq d$ for all separable states ρ on \mathcal{H} , then $n = 2$ and either $d_1 = d_2 = 3$ or $d_2 = 2$.*

Proof. By the theorem we have $l \leq d$ and Eq. (4) implies that $d \leq 1 + 2 \sum_i (d_i - 1)$. Suppose that $n > 2$. Then the function $f(d_1, \dots, d_n) = \prod_i d_i - 1 - 2 \sum_i (d_i - 1)$ is strictly increasing as a function of a single variable d_i . As $f(2, \dots, 2) = 2^n - 2n - 1 > 0$, we have a contradiction. We conclude that $n = 2$, and the corollary follows easily. \square

On the bipartite system $M \otimes N$, if $M = 1$ or $N = 1$, all states are separable and so $\mathcal{S}'_r = \mathcal{S}_r = \mathcal{D}_r$ and $\text{Dim } \mathcal{D}_r = r(2N - r) - 1$ by Theorem 7. In Table I, we give the dimensions of the sets \mathcal{S}'_r for several small values of n and the d_i . As $\mathcal{S}'_1 \subseteq \mathcal{S}'_2 \subseteq \mathcal{S}'_3 \subseteq \dots$, we have $\text{Dim } \mathcal{S}'_1 \leq \text{Dim } \mathcal{S}'_2 \leq \text{Dim } \mathcal{S}'_3 \leq \dots$.

Let us sketch the proof of the results stated in Table I. By Theorem 8 we have $\text{Dim } K\mathcal{S}'_r \leq (2M + 2N - 3)r$. If r is in the initial range (as specified in Table I), we claim that the equality holds. Define the map $\varphi : \mathcal{H}_A \times \mathcal{H}_B \rightarrow H$ by $\varphi(a, b) = |a\rangle\langle a| \otimes |b\rangle\langle b|$ and observe that the rank of its Jacobian matrix does not exceed $2M + 2N - 3$. Next define the map $\Phi_r : (\mathcal{H}_A \times \mathcal{H}_B)^r \rightarrow H$ by setting

$$\Phi_r(a_1, b_1, \dots, a_r, b_r) = \sum_{i=1}^r \varphi(a_i, b_i). \quad (5)$$

The image of Φ_r is exactly the cone $K\mathcal{S}'_r$. Since Φ_r is a smooth map, the dimension of $K\mathcal{S}'_r$ must be greater than or equal to the maximum rank of the Jacobian matrix, $J[\Phi_r]$, of the map Φ_r . Hence, in order to prove the claim it suffices

to find a point $p_r := (a_1, b_1, \dots, a_r, b_r) \in (\mathcal{H}_A \times \mathcal{H}_B)^r$ such that the rank of $J[\Phi_r]$ at p_r is equal to the upper bound $(2M+2N-3)r$. Note that for $r > 1$ we have $J[\Phi_r] = J[\Phi_{r-1} \circ \pi] + J[\phi \circ \pi']$, where $\pi : (\mathcal{H}_A \times \mathcal{H}_B)^r \rightarrow (\mathcal{H}_A \times \mathcal{H}_B)^{r-1}$ and $\pi' : (\mathcal{H}_A \times \mathcal{H}_B)^r \rightarrow \mathcal{H}_A \times \mathcal{H}_B$ are the projection maps sending the point $(a_1, b_1, \dots, a_r, b_r)$ to $(a_1, b_1, \dots, a_{r-1}, b_{r-1})$ and (a_r, b_r) , respectively. Consequently, we have $J[\Phi_r] = J[\Phi_{r-1} \circ \pi] + J[\phi \circ \pi']$ and so

$$\text{rank } J[\Phi_r] - \text{rank } J[\Phi_{r-1} \circ \pi] \leq \text{rank } J[\phi \circ \pi'] \leq 2M + 2N - 3. \quad (6)$$

Thus, it suffices to prove the above claim only for the maximal value, r_{\max} , of r in the initial range. We have done that numerically for all cases in the table by choosing a random point p_r and evaluating the rank of $J[\Phi_r]$ at p_r when $r = r_{\max}$. We used the same method to prove that $\text{Dim } \mathcal{KS}'_l = M^2 N^2 - 1$. The multipartite cases (those with $n > 2$) were treated similarly.

When $M = 2$ and $N > 1$ we have $l = 2N$ and the case $r = l - 1 = 2N - 1$ is exceptional. The inequality from Theorem 8 tells us that $\text{Dim } \mathcal{KS}'_{2N-1} \leq 4N^2 - 1$. We claim that the stronger inequality $\text{Dim } \mathcal{KS}'_{2N-1} \leq 4N^2 - 2$ is valid. Indeed, if $\rho \in \mathcal{KS}'_{2N-1}$ then $L(\rho^\Gamma) = L(\rho) \leq 2N - 1$ and so we must have $\det \rho = 0$ as well as $\det \rho^\Gamma = 0$. Hence, the set \mathcal{KS}'_{2N-1} is contained in each of the two irreducible hypersurfaces $\det \rho = 0$ and $\det \rho^\Gamma = 0$. Consequently, its codimension in H must be at least two. This proves our claim. Note also that if $r = l = 2N$ then the trivial inequality $\text{Dim } \mathcal{KS}'_{2N} \leq 4N^2$ is stronger than the one provided by Theorem 8. In these two cases we again use the ranks of the Jacobian matrices to prove that these improved bounds are attained.

The use of randomness can be avoided with additional effort. In the bipartite case with $M = 2$ and $2 < N < 9$ we set

$$\begin{aligned} |a_k\rangle &= |0\rangle + (k-1)|1\rangle, & |b_k\rangle &= |0\rangle + |k-1\rangle, & k &= 1, \dots, N; \\ |a_k\rangle &= |0\rangle + (k-N)i|1\rangle, & |b_k\rangle &= |0\rangle + |2N-k-1\rangle + |2N-k\rangle, & k &= N+1, \dots, 2N-2; \\ |a_{2N-1}\rangle &= |0\rangle + (N-1)i|1\rangle, & |b_{2N-1}\rangle &= i|1\rangle + |N-1\rangle; \\ |a_{2N}\rangle &= |0\rangle, & |b_{2N}\rangle &= |N-1\rangle, \end{aligned}$$

where i is the imaginary unit. When $N = 2$ we can use the same expressions except that we set $|b_4\rangle = |0\rangle + |1\rangle$. Then one can verify that $J[\Phi_r]$ at the point $p_r := (a_1, b_1, \dots, a_r, b_r)$ has rank $(2N+1)r$ for $r < 2N-1$, and has ranks $4N^2-2$ and $4N^2$ when r is $2N-1$ and $2N$, respectively. Hence, $\text{Dim } \mathcal{KS}'_r$ must be equal to $(2N+1)r$ for $r < 2N-1$, it is $4N^2-2$ for $r = 2N-1$, and $4N^2$ for $r \geq 2N$.

As another example, we consider the bipartite case $M = 3$, $N = 4$ in which case $r_{\max} = 13$. We shall select an explicit point p_{13} where the rank of the Jacobian matrix $J[\Phi_{13}]$ is equal to the upper bound $(2M+2N-3)r_{\max} = 11 \cdot 13 = 143$.

Let us define the following 14 vector pairs $(|a_k\rangle, |b_k\rangle)$ in $3 \otimes 4$:

$$\begin{aligned} |a_1\rangle &= |0\rangle & |b_1\rangle &= |0\rangle \\ |a_2\rangle &= |0\rangle + |1\rangle & |b_2\rangle &= |0\rangle + |1\rangle \\ |a_3\rangle &= |0\rangle - |1\rangle & |b_3\rangle &= |0\rangle + |2\rangle \\ |a_4\rangle &= |0\rangle + i|1\rangle & |b_4\rangle &= |0\rangle - |3\rangle \\ |a_5\rangle &= |0\rangle + |2\rangle & |b_5\rangle &= |0\rangle + |3\rangle \\ |a_6\rangle &= |0\rangle - |2\rangle & |b_6\rangle &= |0\rangle + |1\rangle + |3\rangle \\ |a_7\rangle &= |0\rangle + |1\rangle + |2\rangle & |b_7\rangle &= |0\rangle - |2\rangle \\ |a_8\rangle &= |0\rangle - |1\rangle + |2\rangle & |b_8\rangle &= |0\rangle - i|1\rangle \\ |a_9\rangle &= |0\rangle + (1+i)|2\rangle & |b_9\rangle &= |0\rangle + i|1\rangle \\ |a_{10}\rangle &= |0\rangle + i|1\rangle - |2\rangle & |b_{10}\rangle &= |0\rangle + |1\rangle + |2\rangle \\ |a_{11}\rangle &= |0\rangle + |1\rangle + i|2\rangle & |b_{11}\rangle &= |0\rangle + |1\rangle + i|2\rangle \\ |a_{12}\rangle &= |0\rangle + i|1\rangle + |2\rangle & |b_{12}\rangle &= |0\rangle + i|2\rangle + |3\rangle \\ |a_{13}\rangle &= |0\rangle + i|1\rangle + i|2\rangle & |b_{13}\rangle &= |0\rangle + |2\rangle + |3\rangle \\ |a_{14}\rangle &= |0\rangle - i|1\rangle & |b_{14}\rangle &= |0\rangle - |1\rangle. \end{aligned}$$

Set $p_r := (a_1, b_1, \dots, a_r, b_r)$ for $r = 1, \dots, 14$. We have verified that the rank of $J[\Phi_{13}]$ at the point p_{13} is 143, and the rank of $J[\Phi_{14}]$ at the point p_{14} is 144. Since $\text{Dim } \mathcal{KS}'_{12} = 132$, each neighborhood of the point $\Phi_{13}(p_{13})$ must contain infinitely many separable states of length 13. A similar property is shared by the point $\Phi_{14}(p_{14})$. However, it remains unclear whether the state $\Phi_{13}(p_{13})$ has length 13, and the state $\Phi_{14}(p_{14})$ the length 14.

It is tempting to conjecture that the equality $\text{Dim } \mathcal{S}'_r = \text{Dim } \mathcal{S}'_{r+1}$ implies that $\mathcal{S}'_r = \mathcal{S}'_{r+1}$. However, it was shown very recently that when $M = N = 3$ there exist separable states of length 10. Hence, in this case $\mathcal{S}'_9 \subset \mathcal{S}'_{10}$ while $\text{Dim } \mathcal{S}'_9 = \text{Dim } \mathcal{S}'_{10} = 80$. In view of this example, we shall propose a conjecture only for bipartite systems with $M = 2$ and N arbitrary, in which case $l = 2N$.

Conjecture 10 *Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be the Hilbert space of a bipartite quantum system. If $\text{Dim } \mathcal{H}_A = 2$ and $\text{Dim } \mathcal{H}_B = N$, then for any separable state ρ on \mathcal{H} we have $L(\rho) \leq 2N$.*

TABLE I: The dimensions of the sets \mathcal{S}'_r of separable states of length at most r in $d_1 \otimes \cdots \otimes d_n$ system. The dimension increases with r and reaches the maximum value $d^2 - 1$ for $r = l$, the bound defined by Eq. (4), and remains constant afterwards.

d_1, \dots, d_n	Initial range	Exceptional case	$d^2 - 1; r \geq l$
2,1	2; $r = 1$		3; $r \geq 2$
2, N ; ($1 < N < 9$)	$(2N + 1)r - 1$; $r < 2N - 1$	$4N^2 - 3$; $r = 2N - 1$	$4N^2 - 1$; $r \geq 2N$
3,3	$9r - 1$; $r < 9$		80; $r \geq 9$
3,4	$11r - 1$; $r < 14$		143; $r \geq 14$
3,5	$13r - 1$; $r < 18$		224; $r \geq 18$
4,4	$13r - 1$; $r < 20$		255; $r \geq 20$
2,2,2	$7r - 1$; $r < 10$		63; $r \geq 10$
2,2,3	$9r - 1$; $r < 16$		143; $r \geq 16$
2,2,4	$11r - 1$; $r < 24$		255; $r \geq 24$
2,2,2,2	$9r - 1$; $r < 29$		255; $r \geq 29$

By Theorem 8, the conjectural bound $2N$ is the best possible. So far, it was known that this conjecture is true in the case of two qubits, but all other cases were open. We have proved recently [5] that it is also true for the qubit-qutrit system. We note that, since a separable state of rank $r \leq 2$ has length r , we have $\mathcal{S}_1 = \mathcal{S}'_1$ and $\mathcal{S}_2 = \mathcal{S}'_2$. On the other hand recall that $\mathcal{S}_d = \mathcal{S}$ contains an open ball of H' centered at the normalized identity matrix $(1/d)I_d$, see [15]. Consequently, $\text{Dim } \mathcal{S}_d = d^2 - 1$. The dimensions of the sets \mathcal{S}_r for $2 < r < d$ are not known. We shall compute them in the two smallest bipartite cases $2 \otimes 2$ and $2 \otimes 3$. The computational method from the proof below can be used to compute the dimensions of the sets \mathcal{P}_r consisting of all bipartite PPT states of rank at most r . (These are semialgebraic sets because $\mathcal{P}_r = \mathcal{D}_r \cap \mathcal{D}^\Gamma$.) Note that in the special cases that we consider here, we have $\mathcal{S}_r = \mathcal{P}_r$ due to the Peres-Horodecki criterion.

Proposition 11 *In $2 \otimes 2$ we have $\text{Dim } \mathcal{S}_3 = 14$. In $2 \otimes 3$ we have $\text{Dim } \mathcal{S}_r = 26, 31, 34$ for $r = 3, 4, 5$, respectively.*

Proof. It is convenient to work with non-normalized states and so we shall use the cones $K\mathcal{S}_r$ and $K\mathcal{D}_r$ instead of \mathcal{S}_r and \mathcal{D}_r . Any non-normalized state ρ of rank at most r can be written as $\rho = C^\dagger C$, where C is an $r \times 2N$ matrix whose diagonal entries are real and those below the diagonal are 0. Let us denote by X the real vector space of such matrices C . Note that $\text{Dim } X = r(4N - r)$. Thus we have a surjective map $g : X \rightarrow K\mathcal{D}_r$ defined by $g(C) = C^\dagger C$. We select a point $C_0 \in X$ such that the state $\rho_0 := g(C_0) \in \mathcal{P}_r$ and, subject to this condition, the rank r_0 of the Jacobian matrix J_0 of g at C_0 is maximal. By continuity, there is an $\epsilon > 0$ such that the rank of the Jacobian matrix is equal to r_0 and $g(C) \in \mathcal{P}_r$ for all $C \in X$ in the open ball $\|C - C_0\| < \epsilon$. The image of this ball is a submanifold of H' of dimension r_0 . Since this image is contained in $K\mathcal{P}_r = K\mathcal{S}_r$, we conclude that $\text{Dim } K\mathcal{S}_r \geq r_0$. Hence, if $r_0 = r(4N - r) = \text{Dim } K\mathcal{D}_r$ we can deduce that $\text{Dim } \mathcal{S}_r = \text{Dim } \mathcal{D}_r$. This is indeed true in the cases when (N, r) is $(2, 3)$, $(3, 3)$, $(3, 4)$ or $(3, 5)$. The matrices C_0 for these four cases can be chosen as follows:

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad [I_3 \ I_3], \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \quad (7)$$

□

We conclude this section with an interesting example.

Example 12 Consider the separable $M \times N$ state $\sigma = \sum_{i=1}^M |i\rangle\langle i| \otimes \rho_i$ with the $\rho_i > 0$. Evidently, $L(\sigma) = \text{rank } \sigma = MN$. If $|a, b\rangle$ is a product vector and $\rho = \sigma + |a, b\rangle\langle a, b|$, then $\text{rank } \rho = MN$ and we claim that $L(\rho) = MN$. There is a unique $p_i > 0$ such that $\sigma_i := \rho_i - p_i |b\rangle\langle b| \geq 0$ and $\text{rank } \sigma_i = N - 1$. Then

$$\rho = \sum_{i=1}^M |i\rangle\langle i| \otimes \sigma_i + \left(|a\rangle\langle a| + \sum_{i=1}^M p_i |i\rangle\langle i| \right) \otimes |b\rangle\langle b| \quad (8)$$

shows that $L(\rho) \leq MN$, and the claim follows. □

An interesting problem is to characterize the separable states σ , say on $M \otimes N$, such that, for every product vector $|a, b\rangle$ on $M \otimes N$, the state $\rho = \sigma + |a, b\rangle\langle a, b|$ satisfies the inequality $L(\rho) \leq L(\sigma)$. The cases where σ has the maximal possible length are of course trivial. The above example shows that there exist σ with the above stated property which are not of this trivial type. Since $L(\rho) \geq \text{rank } \rho$, we infer that in Example 12 we always have $L(\rho) = L(\sigma) = MN$.

We claim that there exist ρ and σ as above such that $L(\rho) < L(\sigma)$. First, we show that the sum of two separable states may have smaller length than one of the two summands. Assume that $(M-2)(N-2) > 1$. Then by Theorem 8 we can choose a separable state σ such that $L(\sigma) > MN$. We can choose large $t > 0$ such that $\rho = tI - \sigma \geq 0$ is separable. Then $L(\rho + \sigma) = MN < L(\sigma)$. Second, let $k = L(\rho)$ and choose a decomposition $\rho = \sum_{i=1}^k |a_i, b_i\rangle\langle a_i, b_i|$. Let $\sigma_r = \sigma + |a_1, b_1\rangle\langle a_1, b_1| + \dots + |a_r, b_r\rangle\langle a_r, b_r|$ for $r = 0, 1, \dots, k$. Since $L(\sigma_k) < L(\sigma)$, there exists an r ($0 < r \leq k$) such that $L(\sigma_r) < L(\sigma_{r-1})$. This proves our claim. For analytic examples of such states in $3 \otimes 3$ see the very recent preprint [10].

However, it is not known whether in $2 \otimes N$ there exists a separable state σ and a state $\rho = \sigma + |a, b\rangle\langle a, b|$ such that $L(\rho) < L(\sigma)$. If such states exist, then

$$L(\sigma) > L(\rho) \geq \max(\text{rank } \rho, \text{rank } \rho^\Gamma) \geq \max(\text{rank } \sigma, \text{rank } \sigma^\Gamma). \quad (9)$$

One can show that there exists a separable state σ whose length is larger than $\max(\text{rank } \sigma, \text{rank } \sigma^\Gamma)$. Actually, analytic examples of such 3×3 states can be found in [9]. However, no such example is known in $2 \otimes N$. If such states do not exist, then it would follow that Conjecture 10 is true.

IV. REAL SEPARABLE STATES

Many important states used in quantum information are *real* i.e., all entries of their density matrices are real. For this to make sense, we have to assume that we have fixed an o.n. basis in each of the Hilbert spaces \mathcal{H}_i , $i = 1, \dots, n$. In this section we shall study the real separable states on \mathcal{H} . We have defined in the Introduction the partial transposition operators Γ_i $i = 1, \dots, n$, as well as Γ_S for $S \subseteq \{1, \dots, n\}$, and the group G . Recall that H is the space of all Hermitian operators on \mathcal{H} , and H' its affine subspace consisting of operators of trace 1.

We say that an operator $\rho \in H$ is *G-invariant*, and we write $\rho^G = \rho$, if each element of G fixes ρ . (This is the case if and only if each Γ_i fixes ρ .) We introduce the following real spaces

$$H^{\text{re}} = \{\rho \in H : \rho^* = \rho\}, \quad (10)$$

$$H^G = \{\rho \in H : \rho^G = \rho\}, \quad (11)$$

$$H'^G = \{\rho \in H' : \rho^G = \rho\}. \quad (12)$$

Thus H^{re} is the space of all real symmetric operators on \mathcal{H} . It is easy to see that $H^G \subseteq H^{\text{re}}$ and that H^G can be identified with the tensor product over \mathbf{R} of the spaces of real symmetric operators on \mathcal{H}_i , $i = 1, \dots, n$. In particular, it follows that

$$\text{Dim } H^G = \prod_{i=1}^n \binom{d_i + 1}{2}. \quad (13)$$

For convenience, we also set

$$\mathcal{D}^{\text{re}} = \mathcal{D} \cap H^{\text{re}}, \quad \mathcal{S}^{\text{re}} = \mathcal{S} \cap H^{\text{re}}, \quad \mathcal{S}^G = \mathcal{S} \cap H^G. \quad (14)$$

As a subset of \mathcal{S} , the set \mathcal{S}^G is defined by the equations $\rho^{\Gamma_i} = \rho$, $i = 1, \dots, n$.

Let $|a_i\rangle \in \mathcal{H}_i$, $i = 1, \dots, n$, be nonzero real vectors and $|\phi\rangle = |a_1, \dots, a_n\rangle$ the corresponding real product vector. Then we say that $|\phi\rangle\langle\phi|$ is a *real pure product state*. These states are obviously G -invariant, i.e., $|\phi\rangle\langle\phi|^{\Gamma_S} = |\phi\rangle\langle\phi|$ for all S . We say that a state ρ is *separable over \mathbf{R}* if it belongs to the convex hull of the set of real pure product states. Consequently, if a separable state ρ is separable over \mathbf{R} , then we must have $\rho^G = \rho$.

For instance, for the two-qubit real separable state $\rho = |00\rangle\langle 00| + |11\rangle\langle 11| + |\psi\rangle\langle\psi|$, with $|\psi\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$, we have $\rho^\Gamma \neq \rho$. Therefore ρ is not separable over \mathbf{R} . For an explicit expression of ρ as the sum of four (complex) pure product states see [1, Eq. (137)].

Our first result is that the above necessary condition is also sufficient.

Proposition 13 *Let ρ be a separable state and $l = L(\rho)$. If $\rho^G = \rho$ then ρ is a sum of 2^{nl} real pure product states, and so ρ is separable over \mathbf{R} . Consequently, \mathcal{S}^G is the set of all states which are separable over \mathbf{R} .*

Proof. We have $\rho = \sum_{k=1}^l |\psi_k\rangle\langle\psi_k|$, where $|\psi_k\rangle = |a_{k1}, \dots, a_{kn}\rangle$. We define

$$|b_{kj}\rangle = (|a_{kj}\rangle + |a_{kj}^*\rangle)/\sqrt{2}, \quad |c_{kj}\rangle = i(|a_{kj}\rangle - |a_{kj}^*\rangle)/\sqrt{2}. \quad (15)$$

It is easy to verify that $|a_{kj}\rangle\langle a_{kj}| + |a_{kj}^*\rangle\langle a_{kj}^*| = |b_{kj}\rangle\langle b_{kj}| + |c_{kj}\rangle\langle c_{kj}|$. Since $\rho^{\Gamma_S} = \rho$ for all $S \subseteq \{1, \dots, n\}$, we have

$$\begin{aligned} 2^n \rho &= \sum_S \sum_{k=1}^l |a_{k1}, \dots, a_{kn}\rangle\langle a_{k1}, \dots, a_{kn}|^{\Gamma_S} \\ &= \sum_{k=1}^l \left(|a_{k1}\rangle\langle a_{k1}| + |a_{k1}^*\rangle\langle a_{k1}^*| \right) \otimes \dots \otimes \left(|a_{kn}\rangle\langle a_{kn}| + |a_{kn}^*\rangle\langle a_{kn}^*| \right) \\ &= \sum_{k=1}^l \left(|b_{k1}\rangle\langle b_{k1}| + |c_{k1}\rangle\langle c_{k1}| \right) \otimes \dots \otimes \left(|b_{kn}\rangle\langle b_{kn}| + |c_{kn}\rangle\langle c_{kn}| \right). \end{aligned} \quad (16)$$

Since $|b_{kj}\rangle$ and $|c_{kj}\rangle$ are real, this completes the proof. \square

For $\rho \in K\mathcal{S}^G$ we denote by $L^G(\rho)$ the smallest integer k such that ρ can be written as a sum of k real pure product states. We also say that $L^G(\rho)$ is the *length of ρ over \mathbf{R}* . Finally, for any positive integer r , we set $\mathcal{S}_r^G = \{\rho \in \mathcal{S}^G : L^G(\rho) \leq r\}$. It is immediate from the definitions that the sets \mathcal{S}^{re} , \mathcal{S}^G and \mathcal{S}_r^G are semialgebraic.

In general, for $\rho \in \mathcal{S}^G$ we have $L(\rho) \leq L^G(\rho)$. It is an open question whether the equality always holds. E.g., we do not know whether ρ in Proposition 13 is a sum of l real pure product states. The next theorem shows that the equality $L(\rho) = L^G(\rho)$ holds in any $2 \otimes N$ system.

It was shown in [13, Theorem 2] that a $2 \times N$ PPT state ρ with $\rho^\Gamma = \rho$ is separable. Moreover, such ρ admits a decomposition $\rho = \sum_{i=1}^r |a_i, b_i\rangle\langle a_i, b_i|$ with $r = \text{rank } \rho$ and all $|a_i\rangle$ real. For real ρ we have the following stronger version of this result.

Theorem 14 *In any $2 \otimes N$ system, every G -invariant state ρ is separable over \mathbf{R} . Moreover, for such ρ we have $L^G(\rho) = \text{rank } \rho$.*

Proof. To prove the first assertion, it suffices to consider the case where ρ is a $2 \times N$ state. By the theorem cited above, ρ is separable. Thus, $\rho \in \mathcal{S}^G$ and the assertion follows from Proposition 13.

To prove the second assertion, we again may assume that ρ is a $2 \times N$ state. It follows that $r := \text{rank } \rho \geq N$. Since $\rho^\Gamma = \rho$, we have

$$\rho = \begin{bmatrix} A & B \\ B & C \end{bmatrix}, \quad (17)$$

where A, B, C are real symmetric matrices of order N . Moreover, we can assume that C is invertible (see e.g. [4, Example 2]). By performing an invertible real local operation on \mathcal{H}_B , we can assume that $C = I_N$. By performing yet another local operation on \mathcal{H}_B , this time with a real orthogonal matrix, we may also assume that B is a diagonal matrix, say $B = \text{diag}(b_0, \dots, b_{N-1})$. Then we have

$$\rho = \sum_{i=0}^{N-1} |\phi_i\rangle\langle\phi_i| + |0\rangle\langle 0| \otimes A', \quad (18)$$

where $|\phi_i\rangle = (b_i|0\rangle + |1\rangle) \otimes |i\rangle$ and $A' = A - B^2$. Since $\rho \geq 0$, we must have $A' \geq 0$. Moreover, it is clear that $r = N + r'$ where $r' = \text{rank } A'$. Since A' is a sum of r' positive semidefinite matrices of rank one, and the $|\phi_i\rangle$ are real product vectors, the assertion is proved. \square

The first assertion of this theorem may fail for some other quantum systems. To construct a 3×3 counter-example, we start with an UPB $\{|a_i, b_i\rangle : i = 1, \dots, 5\}$ consisting of real product vectors. Then the state $\rho = I - \sum_{i=1}^5 |a_i, b_i\rangle\langle a_i, b_i|$ is G -invariant and entangled [6]. One can construct similarly a $2 \times 2 \times 2$ counter-example by using the UPB from [6, Eq. (22)].

We can compute the dimensions of \mathcal{S}^{re} and \mathcal{S}^G .

Proposition 15 *We have $\text{Dim } \mathcal{S}^{\text{re}} = \binom{d+1}{2} - 1$ and $\text{Dim } \mathcal{S}^G = \prod_i \binom{d_i+1}{2} - 1$.*

TABLE II: Lower bounds for the dimensions of the sets \mathcal{S}_r^G ($r = 1, 2, \dots, d$) of G -invariant real separable states ρ with $L^G(\rho) \leq r$ in $M \otimes N$ systems with $d = MN \leq 16$.

$M, N \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2, 1	1	2														
2, 2	2	5	7	8												
2, 3	3	7	11	14	16	17										
2, 4	4	9	14	19	23	26	28	29								
2, 5	5	11	17	23	29	34	38	41	43	44						
2, 6	6	13	20	27	34	41	47	52	56	59	61	62				
2, 7	7	15	23	31	39	47	53	62	68	73	77	80	82	83		
2, 8	8	17	26	35	44	53	62	71	79	86	92	97	101	104	106	107
3, 1	2	4	5													
3, 2	3	7	11	14	16	17										
3, 3	4	9	14	19	24	29	32	34	35							
3, 4	5	11	17	23	29	35	41	47	53	56	58	59				
3, 5	6	13	20	27	34	41	48	55	62	69	76	83	86	88	89	
4, 1	3	6	8	9												
4, 2	4	9	14	19	23	26	28	29								
4, 3	5	11	17	23	29	35	41	47	53	56	58	59				
4, 4	6	13	20	27	34	41	48	55	62	69	76	83	90	96	98	99

Proof. Recall that there is an open ball, say B , in H' centered at the state $\rho_0 := (1/d)I_d$ such that $B \subseteq \mathcal{S}$. Hence, the first formula follows from the facts that $\rho_0 \in H'^{\text{re}}$ and $\text{Dim } H'^{\text{re}} = \binom{d+1}{2} - 1$, where $H'^{\text{re}} = H' \cap H^{\text{re}}$. The second formula follows from Eq. (13) by a similar argument. \square

In Table II we exhibit the lower bounds for the dimensions of the sets \mathcal{S}_r^G for several bipartite systems of small dimension ($d \leq 16$). In most cases we have proved that these bounds are equal to $\text{Dim } \mathcal{S}_r^G$. To be precise, we know that the last number (for $r = d$) of each item is correct since it is equal to the dimension of \mathcal{S}^G as given by Proposition 15. The first number (for $r = 1$) is also correct, it is equal to $M + N - 2$. The numbers following it are correct as long as the difference between the consecutive numbers is equal to $M + N - 1$. This follows from the fact that $\text{Dim } \mathcal{S}_{r+1}^G - \text{Dim } \mathcal{S}_r^G \leq M + N - 1$ for each r . (This inequality can be proved by a method similar to one we used to prove (6).)

Let us state the general separability problem for arbitrary multipartite systems $\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$ and its two special cases.

- (S_1) For $\rho \in \mathcal{D}$, decide whether $\rho \in \mathcal{S}$.
- (S_2) For $\rho \in \mathcal{D} \cap H^{\text{re}}$, decide whether $\rho \in \mathcal{S}$.
- (S_3) For $\rho \in \mathcal{D} \cap H^G$, decide whether $\rho \in \mathcal{S}$.

Since $H^G \subseteq H^{\text{re}}$, (S_3) is a special case of (S_2). Next we show that (S_1) can be reduced to (S_3) at the expense of enlarging the dimension of the quantum system. So the three problems in (S_1), (S_2) and (S_3) are indeed equivalent. Since NPT states are entangled, it suffices to consider the PPT states only.

Proposition 16 *Assume that each party A_i consists of two parties: party $A_{1,i}$ which has a qubit and another party $A_{2,i}$. So, we have $\mathcal{H}_i = \mathcal{H}_{1,i} \otimes \mathcal{H}_{2,i}$. Let ρ be a PPT state on $\mathcal{H}_{2,1} \otimes \dots \otimes \mathcal{H}_{2,n}$. Then there exists a state σ on the system \mathcal{H} , which belongs to H^G , and is such that ρ is separable if and only if σ is separable.*

Proof. The partial transposition operator Γ_i is the product of the transposition operators Γ'_i and Γ''_i on $\mathcal{H}_{1,i}$ and $\mathcal{H}_{2,i}$, respectively. Similarly, for any subset $S \subseteq \{1, \dots, n\}$, we have $\Gamma_S = \Gamma'_S \Gamma''_S$. Let us define the Hermitian operator σ , acting on the bipartite composite system $\mathcal{H} = \mathcal{H}_{A_{1,1}, \dots, A_{1,n}} \otimes \mathcal{H}_{A_{2,1}, \dots, A_{2,n}}$,

$$\sigma = \sum_S \tau^{\Gamma'_S} \otimes \rho^{\Gamma''_S}, \quad (19)$$

where $\tau = |a, \dots, a\rangle\langle a, \dots, a|_{A_{1,1}, \dots, A_{1,n}}$, $|a\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, and the summation is over all subsets S of the set $\{1, \dots, n\}$. Since ρ is PPT, we have $\sigma \geq 0$. If $R \subseteq \{1, \dots, n\}$, then $\Gamma_R \Gamma_S = \Gamma_{R \Delta S}$ where $R \Delta S := (R \setminus S) \cup (S \setminus R)$ is

the symmetric difference of R and S . Consequently, we have

$$\sigma^{\Gamma_R} = \sum_S \tau^{\Gamma'_{R\Delta S}} \otimes \rho^{\Gamma''_{R\Delta S}} = \sigma. \quad (20)$$

The last equality holds because when S runs through all subsets of $\{1, \dots, n\}$ so does $R\Delta S$. Thus we have shown that $\sigma \in H^G$. Since $\langle a|a^* \rangle = 0$, we have $\rho = \langle a, \dots, a|\sigma|a, \dots, a \rangle$. This formula and Eq. (19) show that ρ is separable if and only if σ is separable. \square

V. CONCLUSIONS

It is well known that the set of all normalized states, \mathcal{D} , has nonempty interior when viewed as a subset of the ambient affine space H' , and so $\text{Dim } \mathcal{D} = d^2 - 1$. No such result is known for its subsets $\mathcal{D}_r = \{\rho \in \mathcal{D} : \text{rank } \rho \leq r\}$. First of all, we have shown that \mathcal{D}_r are real semialgebraic sets and so they have a well defined dimension. Then we have given a simple formula for their dimensions (see Theorem 7).

Next consider the set, \mathcal{S} , of normalized separable states. First we dealt with its subsets $\mathcal{S}'_r = \{\rho \in \mathcal{S} : L(\rho) \leq r\}$, where $L(\rho)$ is the length of ρ . We showed that each \mathcal{S}'_r is semialgebraic and deduced from this fact that \mathcal{S} itself is semialgebraic. We have obtained in Theorem 8 very good upper bounds for $\text{Dim } \mathcal{S}'_r$. These bounds are not of interest when they exceed $d^2 - 1$. However, in most (but not all) of the other cases that we have computed (see Table I) these bounds are saturated. A simple consequence of these bounds is the fact that there exist separable states ρ with $L(\rho) > d := \text{Dim } \mathcal{H}$. The dimension of the subset $\mathcal{S}_r = \{\rho \in \mathcal{S} : \text{rank } \rho \leq r\}$, is much harder to compute. We have done that for the systems $2 \otimes 2$ and $2 \otimes 3$ only (see Proposition 11).

We have initiated the study of real separable states. It may be surprising that such states are not necessarily separable over \mathbf{R} , i.e., not necessarily expressible as a sum of real pure product states (see the example above Proposition 13). On the other hand we show that, among all separable states, those which are separable over \mathbf{R} are characterized by the property of being G -invariant. In addition to the standard separability problem (S_1), we have formulated in section IV two variations (S_2) and (S_3) which ask to decide whether a state ρ belongs to the set of real separable states \mathcal{S}^{re} or the set of G -invariant separable states \mathcal{S}^G , respectively. We have shown that all three separability problems are equivalent to each other.

Last, but not least, we have proposed a method of solving the standard separability problem (S_1) in some low-dimensional quantum systems (see Conjectures 4 and 5). Since these very natural conjectures are valid in the two cases where the separability problem has been solved, namely $2 \otimes 2$ and $2 \otimes 3$, we are hopeful that they may lead to eventual analytic solution of the problem in some additional cases. One possibility is to use the Jacobian matrix of the map Φ_r , see Eq. (5), where r is chosen so that $\mathcal{S}'_r = \mathcal{S}$. In that case, this matrix must have deficient rank at the points mapped to the boundary of the cone $K\mathcal{S}$. This may help us to find the polynomial equations defining the boundary of \mathcal{S} .

The separability problem (S_3) is the real analog of the standard separability problem (S_1). Due to the fact that the set \mathcal{S}^G has much lower dimension than \mathcal{S} , it is very likely that in concrete cases it would be much easier to solve (S_3) than (S_1). As we have shown that (S_3) has a very simple solution for $2 \otimes N$ systems (see Theorem 14), the smallest open cases are $2 \otimes 2 \otimes 2$ and $3 \otimes 3$.

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